Scattering on effective strings and compactified membranes

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Based on 2308.12189 with A. Tseytlin

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- Understand confinement in QCD
- Lattice results for QCD in $D = 3, D = 4$
- What is the worldsheet theory of confining strings?
- How are they different from fundamental strings?

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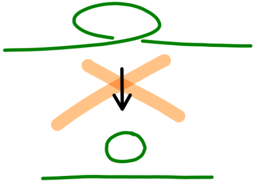

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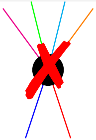
- In this talk: consider the membrane action ($d = 3$) instead of Nambu-Goto string ($d = 2$)

[Zamolodchikov Zamolodchikov '79]

- No particle production 

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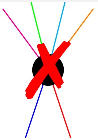
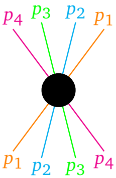


- Transmitted momenta



Worldsheet S matrix – Integrable S matrices

[Zamolodchikov Zamolodchikov '79]

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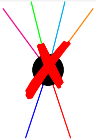

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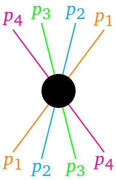
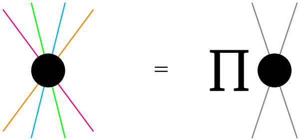
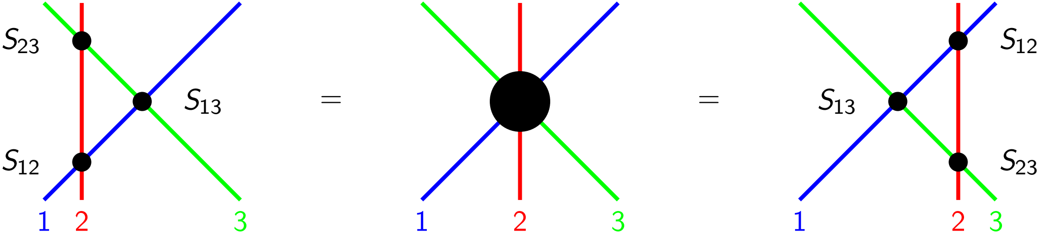


- Factorisation



[for massive scattering]

- Two-body S-matrix satisfies the Yang-Baxter equation



$$S_{23}\,S_{13}\,S_{12} = S_{12}\,S_{13}\,S_{23}$$

- In perturbation theory

$$\mathcal{S} = 1 + i\mathcal{T} = 1 + ig\mathcal{T}^{(0)} - g^2\mathcal{T}^{(1)} + \dots$$

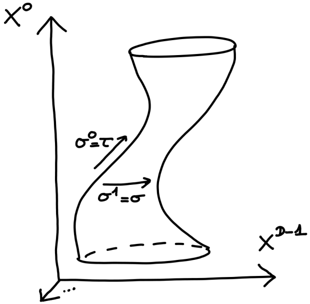
- At tree-level satisfies the classical Yang-Baxter equation

$$[\mathcal{T}_{12}, \mathcal{T}_{13}] + [\mathcal{T}_{12}, \mathcal{T}_{23}] + [\mathcal{T}_{13}, \mathcal{T}_{23}] = 0$$

- Nambu-Goto-Dirac action

$$S = -T_{d-1} \int d^d\sigma \sqrt{-\det \gamma_{\alpha\beta}}\ , \qquad \gamma_{\alpha\beta} = \eta_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu\ .$$

  - Worldsheet coordinates σ^α with $\alpha = 0, 1, \ldots, d-1$
  - Target space coordinates X^μ with $\mu = 0, 1, \ldots, D-1$



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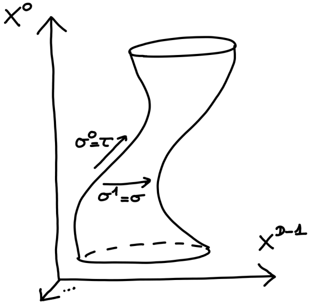
- Interested in string ($d = 2$) and membrane ($d = 3$), arbitrary D



- Expand action near infinite string/brane vacuum: static gauge [Dubovsky Flauger Gorbenko '12]

$$X^0 = \sigma^0 \,, \qquad X^{D-d+1} = \sigma^1 \,, \qquad \ldots \qquad X^{D-1} = \sigma^{d-1}$$

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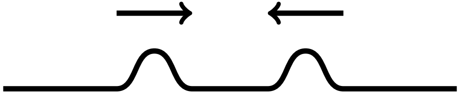
$$X^j \,, \qquad j = 1, \ldots, \hat{D} \equiv D - d$$



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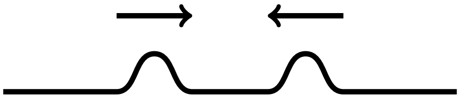
- Expansion of the action

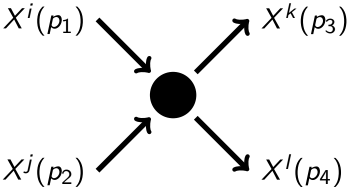


$$S = -T_d \int d^d\sigma \, (1 + \mathcal{L}_2 + \mathcal{L}_4 + \ldots)$$

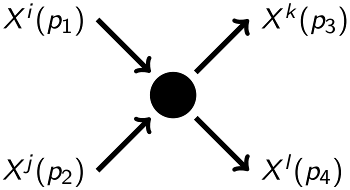
$$\mathcal{L}_2 = \frac{1}{2} \partial_\alpha X^j \partial^\alpha X^j$$

$$\mathcal{L}_4 = \frac{1}{4} \Big(c_2 (\partial_\alpha X^j \partial^\alpha X^j)^2 + c_3 (\partial_\alpha X^j \partial^\beta X^j \partial_\beta X^k \partial^\alpha X^k) \Big) \,, \qquad c_2 = \frac{1}{2} \,, \qquad c_3 = -1$$

4-point scattering



$$s = -(p_1 + p_2)^2$$
$$t = -(p_1 - p_3)^2$$
$$u = -(p_1 - p_4)^2$$



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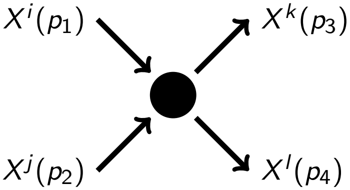
- Scattering amplitude

$$\mathcal{M}^{ij,kl}[s,t,u] = \left(A\,\delta^{ij}\delta^{kl} + B\,\delta^{ik}\delta^{jl} + C\,\delta^{il}\delta^{jk}\right)\delta^{(d)}(p_1 + p_2 - p_3 - p_4)$$



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- Crossing relations

$$B = A|_{s\leftrightarrow t}\ , \qquad C = A|_{s\leftrightarrow u}$$

- Tree-level amplitudes

$$A^{(0)} = \frac{1}{4}(2c_2+c_3)s^2-\frac{1}{2}c_3 tu\,, \quad B^{(0)} = \frac{1}{4}(2c_2+c_3)t^2-\frac{1}{2}c_3 su\,, \quad C^{(0)} = \frac{1}{4}(2c_2+c_3)u^2-\frac{1}{2}c_3 st\,.$$

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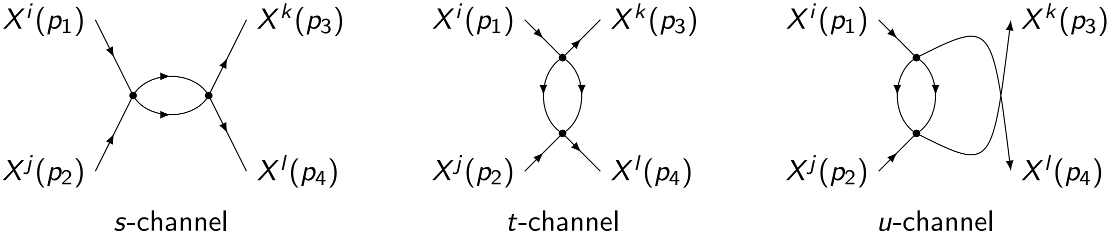
- Amplitude proportional to identity

- Bubble diagrams



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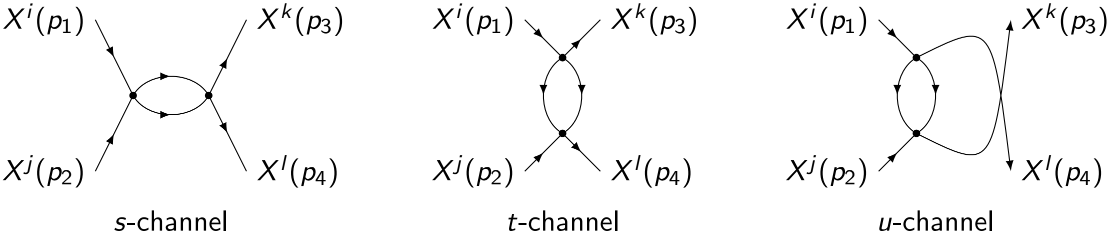

- Specialise to NG coefficients, use dimensional regularisation $d = 2 - 2\epsilon$.

$$A_{\epsilon}^{(1)} = -\frac{\hat{D}-6}{96\pi}\left(\frac{1}{\epsilon} - \gamma + \ln 4\pi\right) stu \ ,$$

$$A_{f}^{(1)} = -\frac{1}{192\pi}\left[(\hat{D}-24)s^3 + \left(\frac{16}{3}\hat{D} + 12 - 2(\hat{D}-6)\log\frac{-s}{\mu^2}\right) stu + 12\left(t\log\frac{s}{t} + u\log\frac{s}{u}\right) tu\right].$$

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  - divergent part vanishes for all amplitudes
  - First term proportional to $\hat{D} - 24 = D - 26$ vanishes in critical dimension
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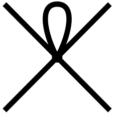

4-point scattering: One-loop amplitude

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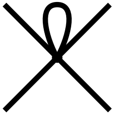

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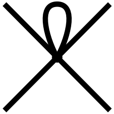

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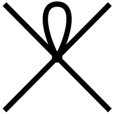

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  [Rosenhaus Smolkin '19]

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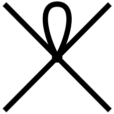

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4-point scattering: One-loop amplitude

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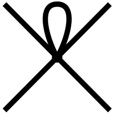

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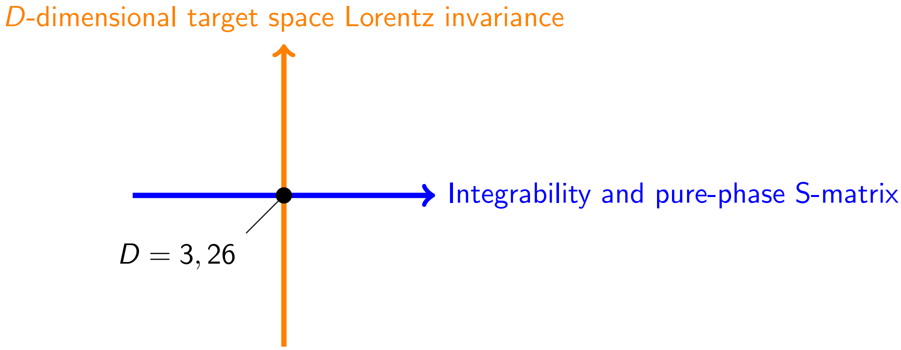
- Actually true to all-loop order

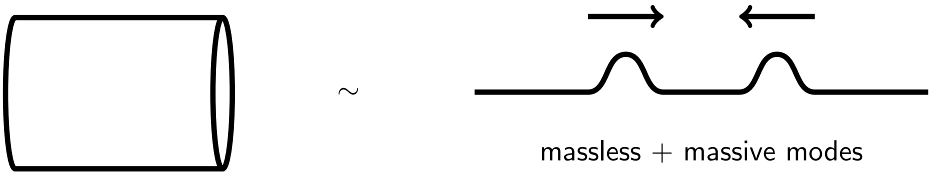
$$S = e^{\frac{i}{4}s}$$


D-dimensional target space Lorentz invariance
Integrability and pure-phase S-matrix
$D = 3, 26$

- There is no standard notion (factorised scattering) of integrability in 3d.

Membrane

- There is no standard notion (factorised scattering) of integrability in 3d.
- Assume membrane has one compact dimension: expand around $\mathbb{R} \times S^1$ vacuum.



massless + massive modes

- Expansion in Fourier modes

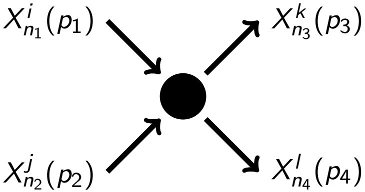
$$X^j(\sigma^0,\sigma^1,\sigma^2) = \sum_{n=-\infty}^{\infty} X_n^j(\sigma^0,\sigma^1)\, e^{\frac{in}{R}\sigma^2}\ , \qquad j=1,\dots,\hat{D}\equiv D-3\ , \qquad X_{-n}^j = (X_n^j)^*\ .$$

- Effective 2d action

$$S = -\hat{T}\int d^2\sigma\left(1+\hat{\mathcal{L}}_2+\hat{\mathcal{L}}_4+\dots\right)\ , \qquad \hat{T}\equiv 2\pi R T_2\ ,$$

$$\hat{\mathcal{L}}_2 = \frac{1}{2}\sum_{n=-\infty}^{\infty}\left(|\partial_\alpha X_n^j|^2 + m_n^2|X_n^j|^2\right) = \frac{1}{2}(\partial_\alpha X_0^j)^2 + \sum_{n=1}^{\infty}\left(|\partial_\alpha X_n^j|^2 + m_n^2|X_n^j|^2\right)\ , \quad m_n = \frac{n}{R}\ ,$$

$$\hat{\mathcal{L}}_4 = \sum_{n_1,\dots,n_4=-\infty}^{\infty}\frac{1}{4}\left(c_2 V^{j,j}_{n_1,n_2}V^{k,k}_{n_3,n_4} + c_3 V^{j,k}_{n_1,n_4}V^{j,k}_{n_2,n_3}\right)\delta_{n_1+\dots+n_4}\ , \quad V^{j,k}_{n_1,n_2}\equiv \partial_\alpha X^j_{n_1}\partial^\alpha X^k_{n_2} - \frac{n_1 n_2}{R^2}X^j_{n_1}X^k_{n_2}\ .$$



$$s + t + u = \sum_j n_j^2$$

$+$ 1 additional constraint for $2d$ scattering

$$A^{(0)} = \frac{1}{4}(2c_2 + c_3)\left[s - (n_1 + n_2)^2\right]^2 - \frac{1}{2}c_3\left[t - (n_1 - n_3)^2\right]\left[u - (n_1 - n_4)^2\right]$$

- Amplitude is not proportional to the identity
- "Particle transmutation", forbidden in an integrable theory. Yang-Baxter not satisfied.
- The compactified membrane theory is already not integrable at tree-level
- Let's push on and compute the one-loop scattering amplitude

- Fixed n calculation, using dimensional regularisation

$$\hat{A}^{(1)}_{n,\epsilon} = \frac{1}{96\pi}\left(\frac{1}{\epsilon} - \gamma + \log 4\pi\right)\Big[-(\hat{D}-6)stu - 12n^2\big(s^2 - \tfrac{1}{2}\hat{D}tu\big)\Big]$$

$$\hat{A}^{(1)}_{n,f} = -\frac{s^2}{192\pi}\Big[(\hat{D}-24)s + 6(\hat{D}+4)n^2 - 24n^2\ln n^2 - 6n^2(\hat{D}n^2 - 2s)Q_n(-s) + 12n^2 sQ_n(s)\Big]\ ,$$

$$\hat{B}^{(1)}_{n,f} = \frac{s^2}{192\pi}\Big[-12\hat{D}n^2 + 12\hat{D}n^2\ln n^2 + 6s(s-2n^2)Q_n(-s) + 6s(s+2n^2)Q_n(s)\Big]\ ,$$

$$\hat{C}^{(1)}_{n,f} = \frac{s^2}{192\pi}\Big[(\hat{D}-24)s - 6(\hat{D}+4)n^2 + 24n^2\ln n^2 + 6n^2(\hat{D}n^2 + 2s)Q_n(s) + 12n^2 sQ_n(-s)\Big]\ .$$

$$Q_n(s) = -\frac{2}{s\sqrt{1+\frac{4n^2}{s}}}\ln\frac{\sqrt{1+\frac{4n^2}{s}}-1}{\sqrt{1+\frac{4n^2}{s}}+1}$$

- Total amplitude

$$\hat{A}^{(1)} = A_0^{(1)} + 2\sum_{n=1}^{\infty} \hat{A}_n^{(1)}$$

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$$\sum_{n=-\infty}^{\infty} n^{2k} = 0 \ , \qquad \sum_{n=-\infty}^{\infty} 1 = 0 \ , \qquad \sum_{n=1}^{\infty} n^2 \ln n^2 = -2\zeta_R'(-2)$$

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- What appears in amplitude is $n^{2k} Q_n(s)$ with $k = 0, 1, 2$. Use asymptotic behaviour

$$Q_n(s)|_{n\to\infty} = \frac{1}{n^2} - \frac{s}{6n^4} + O(\frac{1}{n^6})\ , \qquad Q_n(s)|_{n\to 0} = -\frac{2}{s}\ln\frac{n^2}{s} + O(n^2)$$

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- Finite total amplitude, non-polynomial in s.

- Restore radius of compactification R.

4-point scattering: Limits

- Restore radius of compactification R.
- In the limit $R \to \infty$

$$\hat{A}^{(1)}\Big|_{R\to\infty} = \frac{1}{2\pi R}\frac{1}{256\, T_2^2}\Big[(-s)^{3/2}(\tfrac{3}{32}\hat{D} - 1)s^2 + (s)^{5/2}\Big]$$

  - Same as restriction to $t = 0, u = -s$ of the 3d amplitude

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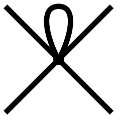
  - ▶ Same as restriction to $t = 0, u = -s$ of the 3d amplitude

- In the limit $R \to 0$

$$\hat{A}^{(1)} = \frac{1}{\hat{T}^2}\Big[\frac{\zeta_R(3)}{8\pi^3 R^2}s^2 - \frac{\hat{D}-24}{192\pi}s^3\Big]$$

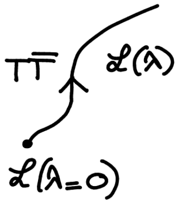


  - ▶ Same as NG, first term cancels when adding tadpole contribution

- One-parameter deformation constructed from the energy-momentum tensor

[Smirnov Zamolodchikov '16 ; Cavaglia Negro Szecsenyi Tateo '16]

$$\partial_\lambda \mathcal{L} = O_{T\bar{T}} \,, \qquad O_{T\bar{T}} = -\det T_{\alpha\beta}$$



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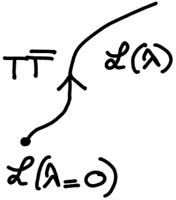

- Infinitesimal deformation

$$\mathcal{L}(\delta\lambda) = \mathcal{L}(\lambda = 0) + O_{T\bar{T}}$$

The $T\bar{T}$ deformation

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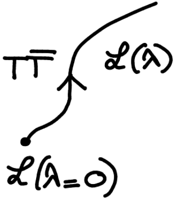

- Infinitesimal deformation

$$\mathcal{L}(\delta\lambda) = \mathcal{L}(\lambda = 0) + O_{T\bar{T}}$$

- Can be applied to any 2d theory with energy-momentum tensor
- When applied to a CFT, theory after deformation may no longer be a CFT
- Irrelevant deformation in terms of RG flow
- $O_{T\bar{T}}$ well-defined quantum mechanically [Zamolodchikov '04]

- Effect on the S-matrix is simple [Zamolodchikov '04]

$$S_\lambda = e^{i\lambda\Phi} S_0$$

- Φ is a CDD [Castillejo Dalitz Dyson '56] factor, constrained by unitarity, crossing
- If S_0 is integrable then so is $S_\lambda \Rightarrow$ theory can be solved exactly

- $T\bar{T}$ of free massless fields gives NG action

$$\mathcal{L}(\lambda = 0) = \frac{1}{2}\partial_\alpha X^j \partial^\alpha X^j \qquad \Rightarrow \qquad \mathcal{L} = \frac{1}{\lambda}\left[\sqrt{-\det(\eta_{\alpha\beta} + \lambda\,\partial_\alpha X^j \partial_\beta X^j)} - 1\right]$$

[Cavaglia Negro Szecsenyi Tateo '16] [Bonelli Doroud Zhu '18] [Sfondrini Baggio '18]

- S-matrix is a simple phase factor

$$S_\lambda = e^{\frac{i}{4}\lambda s}$$

- $T\bar{T}$ of free massive fields

$$\mathcal{L}(\lambda=0)=\frac{1}{2}\sum_{n=-\infty}^{+\infty}\left(\partial_\alpha X_n^k\partial^\alpha X^j_{-n}+m_n^2\,X_n^j X^j_{-n}\right)\ ,\qquad m_n^2=\frac{n^2}{R^2}$$

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- Interaction term

$$\mathcal{L}_4=\frac{1}{4}\sum_{n,q=-\infty}^{\infty}\left(c_2\partial_\alpha X_n^j\partial^\alpha X_{-n}^j\partial_\beta X_q^k\partial^\beta X_{-q}^k+c_3\partial_\alpha X_n^j\partial^\beta X_{-n}^j\partial_\beta X_q^k\partial^\alpha X_{-q}^k+\tilde{c}\, m_n^2 m_q^2 X_n^j X_{-n}^j X_q^k X_{-q}^k\right)$$

$$c_2=\frac{1}{2}\ ,\qquad c_3=-1\ ,\qquad \tilde{c}=\frac{1}{2}\ .$$

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- Two fields with contracted indices have opposite mode number

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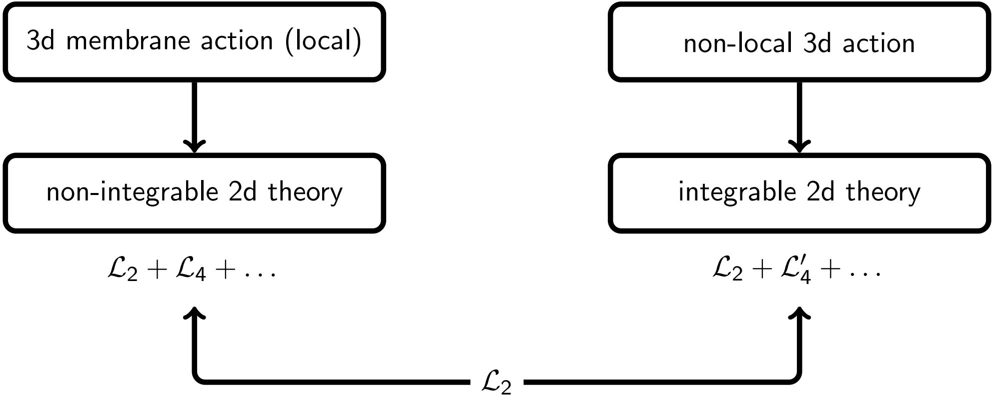
- Two fields with contracted indices have opposite mode number
- Two delta functions on mode numbers $\Rightarrow$ cannot be obtained from a local 3d action


3d membrane action (local)
non-local 3d action
non-integrable 2d theory
integrable 2d theory
$\mathcal{L}_2 + \mathcal{L}_4 + \ldots$
$\mathcal{L}_2 + \mathcal{L}_4' + \ldots$
$\mathcal{L}_2$

- The relation between EFT of confining strings, the Nambu-Goto action and integrable $T\bar{T}$ deformations proved instrumental in obtaining new predictions about flux tube spectrum
- Here looked at generalisation to membrane theory: 3d worldvolume theory
- Compactify additional dimension on circle with radius R: "thick" effective string
- Theory is not integrable already at tree-level
- The limit $R \to 0$ reproduces the 2d NG amplitude.
- The limit $R \to \infty$ reproduces the 3d amplitude.
- Integrable theory with same free spectrum: $T\bar{T}$ deformation of tower of KK modes.

Conclusions and Outlook

- What about superstrings and supermembranes? [Cooper Dubovsky Gorbenko Mohsen Storace '14]
- String/membrane in curved spaces and tests of the AdS_4/CFT_3 duality [Beccaria Giombi Tseytlin '23]
- How to systematically incorporate non-integrable corrections?
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Thank You!